Projective toric designs, difference sets, and quantum state designs

Joseph T. Iosue

arXiv:2311.13479

CodEx Seminar: 13 February 2024

University of Maryland 1856
Joint Center for Quantum Information and Computer Science
NIST
JQI Joint Quantum Institute

Joint work with wonderful collaborators

T. C. Mooney

Adam Ehrenberg

Prof. Alexey Gorshkov

Overview

1. Introduction
2. Complete sets of mutually unbiased bases
3. Bound on minimal projective toric designs
4. Projective toric designs from difference sets
5. Almost minimal quantum state designs
6. Outlook

What is a t-design?

Definition

Given a measure space (M, μ) and a set of polynomials on M, a t-design on M is a measure space $(X \subset M, \nu)$ satisfying $\int_X f \, \mathrm{d}\nu = \int_M f \, \mathrm{d}\mu$ for all polynomials f of degree $f \leq t$.

What is a t-design?

Definition

Given a measure space (M, μ) and a set of polynomials on M, a t-design on M is a measure space $(X \subset M, \nu)$ satisfying $\int_X f \, \mathrm{d}\nu = \int_M f \, \mathrm{d}\mu$ for all polynomials f of degree $f \leq t$.

$M = S^d$	Spherical design	$M = \mathrm{U}(d)$	Unitary design
$M = \Omega_d$	Complex spherical design	$M = \mathrm{PU}(d)$	Projective unitary design
$M = \mathbb{CP}^d$	Quantum state design	$M = \Delta^d$	Simplex design
$M = T^d$	Toric design	$M = P(T^d)$	Projective toric design
$M = S(\mathbb{R})'$	Rigged (continuous variable) quantum state design		

What is a t-design?

Definition

Given a measure space (M, μ) and a set of polynomials on M, a t-design on M is a measure space $(X \subset M, \nu)$ satisfying $\int_X f \, \mathrm{d}\nu = \int_M f \, \mathrm{d}\mu$ for all polynomials f of degree $f \leq t$.

$M = S^d$	Spherical design	$M = \mathrm{U}(d)$	Unitary design
$M = \Omega_d$	Complex spherical design	$M = \mathrm{PU}(d)$	Projective unitary design
$M = \mathbb{CP}^d$	Quantum state design	$M = \Delta^d$	Simplex design
$M = T^d$	Toric design	$M = P(T^d)$	Projective toric design
$M = S(\mathbb{R})'$	Rigged (continuous variable) quantum state design		

What is a t-design?

Definition

Given a measure space (M, μ) and a set of polynomials on M, a t-design on M is a measure space $(X \subset M, \nu)$ satisfying $\int_X f \, \mathrm{d}\nu = \int_M f \, \mathrm{d}\mu$ for all polynomials f of degree $f \leq t$.

$M = S^d$	Spherical design	$M = \mathrm{U}(d)$	Unitary design
$M = \Omega_d$	Complex spherical design	$M = \mathrm{PU}(d)$	Projective unitary design
$M = \mathbb{CP}^d$	Quantum state design	$M = \Delta^d$	Simplex design
$M = T^d$	Toric design	$M = P(T^d)$	Projective toric design
$M = S(\mathbb{R})'$	Rigged (continuous variable) quantum state design		

What is a t-design?

Definition

Given a measure space (M, μ) and a set of polynomials on M, a t-design on M is a measure space $(X \subset M, \nu)$ satisfying $\int_X f \, \mathrm{d}\nu = \int_M f \, \mathrm{d}\mu$ for all polynomials f of degree $f \leq t$.

$M = S^d$	Spherical design	$M = \mathrm{U}(d)$	Unitary design
$M = \Omega_d$	Complex spherical design	$M = \mathrm{PU}(d)$	Projective unitary design
$M = \mathbb{CP}^d$	Quantum state design	$M = \Delta^d$	Simplex design
$M = T^d$	Toric design	$M = P(T^d)$	Projective toric design
$M = S(\mathbb{R})'$	Rigged (continuous variable) quantum state design		

What is a toric design?

Definition

Let $T = \mathbb{R}/2\pi\mathbb{Z}$. A T^n *t-design* (or trigonmetric cubature rule of dimension n and degree t) is a measure space $(X \subset T^n, \nu)$ such that

$$\int_X \exp\left(\mathrm{i}\sum_{j=1}^{n} \alpha_j \phi_j\right) \mathrm{d}\nu(\phi) = \int_{T^n} \exp\left(\mathrm{i}\sum_{j=1}^{n} \alpha_j \phi_j\right) \mathrm{d}\mu_n(\phi)$$

for all $\alpha \in \mathbb{Z}^n$ satisfying $\sum_{j=1}^{n} |\alpha_j| \leq t$, where μ_n is T^n's unit-normalized Haar measure.

A T^n design is the same as a design on the diagonal unitary group $T(\mathrm{U}(n))$.

General theme for projective designs

(Q) What makes a **projective** [complex spherical, toric, unitary] design different from a [complex spherical, toric, unitary] design? **(A)** **The polynomials**

A projective complex spherical design is a complex-projective design

General theme for projective designs

(Q) What makes a **projective** [complex spherical, toric, unitary] design different from a [complex spherical, toric, unitary] design? **(A)** **The polynomials**

Example

On T^2, $\exp(\mathrm{i}(\phi_1 + \phi_2))$ is a degree 2 monomial. But it does *not* descend to a well-defined function on $P(T^2) = T^2/\mathrm{U}(1)$.

A projective complex spherical design is a complex-projective design

What is a projective toric design?

Definition

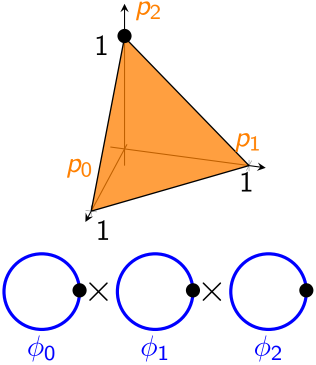
Let $P(T^n) = T^n/\mathrm{U}(1)$. A $P(T^n)$ *t-design* is a measure space $(X \subset P(T^n), \nu)$ such that for all $a, b \in \{1, \ldots, n\}^t$,

$$\int_X \exp\left(\mathrm{i}\sum_{j=1}^{t}(\phi_{a_j} - \phi_{b_j})\right) \mathrm{d}\nu(\phi) = \int_{T^n} \exp\left(\mathrm{i}\sum_{j=1}^{t}(\phi_{a_j} - \phi_{b_j})\right) \mathrm{d}\mu_{n-1}(\phi)$$

where we denote $P(T^n)$'s unit-normalized Haar measure by μ_{n-1} since $P(T^n) \cong T^{n-1}$.

A $P(T^n)$ design is the same as a design on the maximal torus of the projective unitary group $T(\mathrm{PU}(n))$.

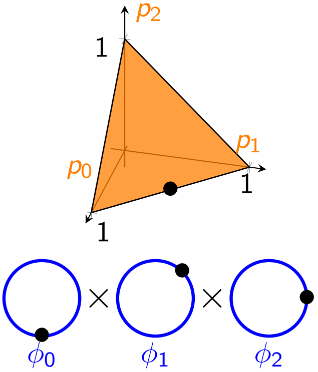
- Consider the parameterization $|p, \phi\rangle := \sum_{n=0}^{d-1} \sqrt{p_n}\, \mathrm{e}^{\mathrm{i}\phi_n} |n\rangle$ of unit vectors in $\mathbb{C}^d$



$$\sqrt{0}\,\mathrm{e}^{\mathrm{i}(0)}|0\rangle + \sqrt{0}\,\mathrm{e}^{\mathrm{i}(0)}|1\rangle + \sqrt{1}\,\mathrm{e}^{\mathrm{i}(0)}|2\rangle$$

Relationship to complex spherical designs

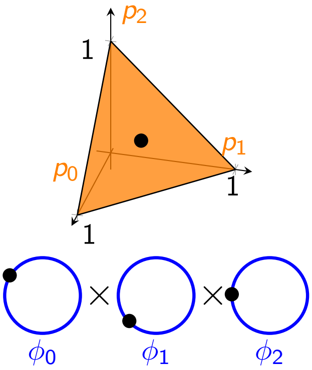
- Consider the parameterization $|p,\phi\rangle := \sum_{n=0}^{d-1} \sqrt{p_n}\, e^{i\phi_n} |n\rangle$ of unit vectors in $\mathbb{C}^d$



$$\sqrt{1/2}\, e^{i(-\pi/2)}|0\rangle + \sqrt{1/2}\, e^{i(\pi/4)}|1\rangle + \sqrt{0}\, e^{i(0)}|2\rangle$$

Relationship to complex spherical designs

- Consider the parameterization $|p, \phi\rangle \coloneqq \sum_{n=0}^{d-1} \sqrt{p_n}\, \mathrm{e}^{\mathrm{i}\phi_n} |n\rangle$ of unit vectors in $\mathbb{C}^d$



$$\sqrt{1/3}\,\mathrm{e}^{\mathrm{i}(5\pi/6)}|0\rangle + \sqrt{1/3}\,\mathrm{e}^{\mathrm{i}(-3\pi/4)}|1\rangle + \sqrt{1/3}\,\mathrm{e}^{\mathrm{i}(\pi)}|2\rangle$$

Relationship to complex spherical designs

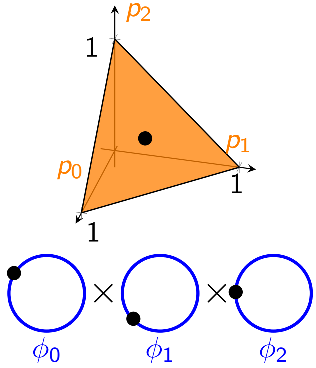
- Consider the parameterization $|p, \phi\rangle := \sum_{n=0}^{d-1} \sqrt{p_n}\, \mathrm{e}^{\mathrm{i}\phi_n} |n\rangle$ of unit vectors in $\mathbb{C}^d$

Theorem

A simplex t-design and a toric t-design combine to yield a complex spherical t-design.

Theorem

A simplex t-design and a **projective** *toric t-design combine to yield a complex* **projective** *t-design.*



$$\sqrt{1/3}\, \mathrm{e}^{\mathrm{i}(5\pi/6)}|0\rangle + \sqrt{1/3}\, \mathrm{e}^{\mathrm{i}(-3\pi/4)}|1\rangle + \sqrt{1/3}\, \mathrm{e}^{\mathrm{i}(\pi)}|2\rangle$$

Relationship to quantum state designs

Fact

Volume integration over Ω_d is equivalent to volume integration over $\Delta^{d-1} \times T^d$

Fact

Volume integration over $\mathbb{CP}^{d-1} = \Omega_d/\mathrm{U}(1)$ is equivalent to volume integration over $\Delta^{d-1} \times P(T^d)$

- Simplex design $\times$ toric design yields complex spherical deisgn
- Simplex design $\times$ projective toric design yields complex projective (quantum state) design

Relationship to quantum state designs

Fact

Volume integration over Ω_d is equivalent to volume integration over $\Delta^{d-1} \times T^d$

Fact

Volume integration over $\mathbb{CP}^{d-1} = \Omega_d/\mathrm{U}(1)$ is equivalent to volume integration over $\Delta^{d-1} \times P(T^d)$

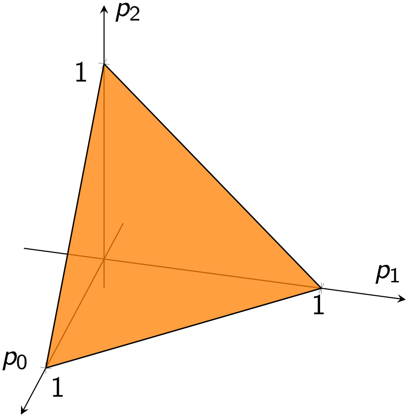
- Simplex design $\times$ toric design yields complex spherical deisgn
- Simplex design $\times$ projective toric design yields complex projective (quantum state) design
- With a suitable redefinition of a “design” on an infinite simplex, one can concatenate such a design with a design on $P(T^\infty)$ to yield a rigged continuous-variable quantum state design (Iosue et al. 2024)

Simplex designs

Definition

The simplex Δ^{d-1} is the set of all probability distributions on d elements

$$\Delta^{d-1} = \left\{ p = (p_0, \ldots, p_{d-1}) \in [0,1]^d \;\middle|\; \sum_{n=0}^{d-1} p_n = 1 \right\}$$



Simplex designs

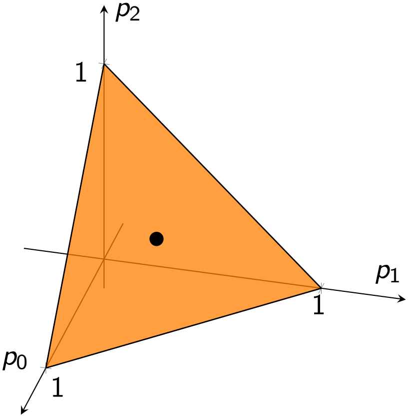
Definition

The simplex Δ^{d-1} is the set of all probability distributions on d elements

$$\Delta^{d-1} = \left\{ p = (p_0, \dots, p_{d-1}) \in [0,1]^d \;\middle|\; \sum_{n=0}^{d-1} p_n = 1 \right\}$$

Example (Simplex 2-design)

The centroid



Simplex designs

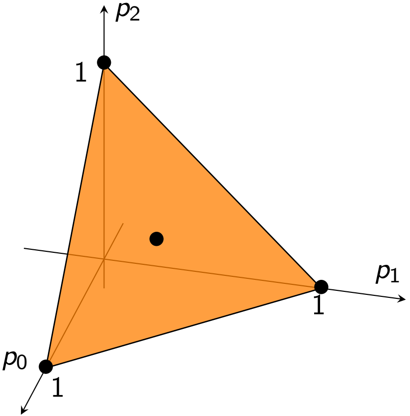
Definition

The simplex Δ^{d-1} is the set of all probability distributions on d elements

$$\Delta^{d-1} = \left\{ p = (p_0, \ldots, p_{d-1}) \in [0,1]^d \;\middle|\; \sum_{n=0}^{d-1} p_n = 1 \right\}$$

Example (Simplex 2-design)

The centroid and the extremal points of the simplex form a 2-design



Overview

Complete set of mutually unbiased bases (CS-MUBs)

Definition

The orthonormal bases $B_0, \ldots, B_d$ of $\mathbb{C}^d$ form a CS-MUBs if $|\langle\psi|\phi\rangle|^2 = 1/d$ for all $\psi \in B_i$ and $\phi \in B_j$ when $i \neq j$.

Complete set of mutually unbiased bases (CS-MUBs)

Definition

The orthonormal bases $B_0, \ldots, B_d$ of $\mathbb{C}^d$ form a CS-MUBs if $|\langle\psi|\phi\rangle|^2 = 1/d$ for all $\psi \in B_i$ and $\phi \in B_j$ when $i \neq j$.

$\iff$

A collection of phases $\{\theta_k^{i,j} \mid i,j,k \in \{1,\ldots,d\}\}$ forms a CS-MUBs if

1. (Orthonormal) $\forall i,j,k: \sum_{\ell=1}^{d} \mathrm{e}^{\mathrm{i}(\theta_\ell^{i,j} - \theta_\ell^{i,k})} = d\delta_{jk}$, and
2. (Mutually unbiased) $\forall i \neq j, k, m: \left|\sum_{\ell=1}^{n} \mathrm{e}^{\mathrm{i}(\theta_\ell^{i,k} - \theta_\ell^{j,m})}\right|^2 = d.$

Each $\theta^{i,j} \in T^d$

Complete set of mutually unbiased bases (CS-MUBs)

Definition

The orthonormal bases $B_0, \ldots, B_d$ of $\mathbb{C}^d$ form a CS-MUBs if $|\langle\psi|\phi\rangle|^2 = 1/d$ for all $\psi \in B_i$ and $\phi \in B_j$ when $i \neq j$.

$\iff$

A collection of phases $\{\theta_k^{i,j} \mid i, j, k \in \{1, \ldots, d\}\}$ forms a CS-MUBs if

1. (Orthonormal)
   $\forall i, j, k: \sum_{\ell=1}^{d} \mathrm{e}^{\mathrm{i}(\theta_\ell^{i,j} - \theta_\ell^{i,k})} = d\delta_{jk}$, and
2. (Mutually unbiased)
   $\forall i \neq j, k, m: \left|\sum_{\ell=1}^{n} \mathrm{e}^{\mathrm{i}(\theta_\ell^{i,k} - \theta_\ell^{j,m})}\right|^2 = d.$

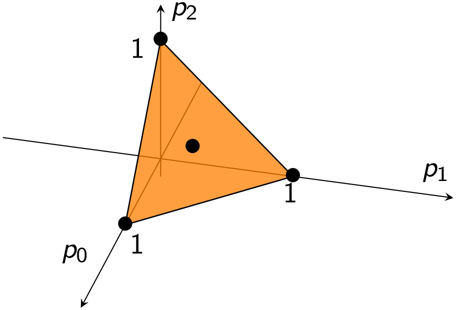
Each $\theta^{i,j} \in T^d$, but overall phase does not matter, so really $\theta^{i,j} \in P(T^d)$

CS-MUBs and projective toric 2-designs

Theorem

A collection $\Theta = \{\theta^{i,j} \mid i,j \in \{1,\dots,d\}\} \subset P(T^d)$ *forms a CS-MUBs iff*

1. *(Orthonormal)* $\forall i,j,k: \sum_{\ell=1}^{d} \mathrm{e}^{\mathrm{i}(\theta_\ell^{i,j} - \theta_\ell^{i,k})} = d\delta_{jk}$, *and*
2. Θ *is a projective toric 2-design.*



CS-MUB example

Let $d = p$ be a prime. Then $\theta_k^{i,j} = \frac{2\pi}{p}(jk + ik^2)$ is a projective toric 2-design and satisfies orthonormality.

$\Updownarrow$ concatenate with simplex design

$B_0, \ldots, B_d$ forms a CS-MUBs for $\mathbb{C}^p$, where $B_0 = \{|j\rangle \mid j \in \{1, \ldots, p\}\}$ and $B_i = \left\{|\psi^{i,j}\rangle = \frac{1}{\sqrt{p}} \sum_{k=1}^{d} \mathrm{e}^{\mathrm{i}\theta_k^{i,j}} |k\rangle \mid j \in \{1, \ldots, p\}\right\}$

This is the canonical example of a CS-MUBs from (Wootters and Fields 1989)

Infinite dimensions

$\left\{\theta^{\varphi,\vartheta} = \left(\vartheta k + \varphi k^2\right)_{k\in\mathbb{N}} \mid \vartheta,\varphi \in [0,2\pi)\right\}$ is a $P(T^\infty)$ 2-design. That is,

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\left(\theta_a^{\varphi,\vartheta}+\theta_b^{\varphi,\vartheta}-\theta_c^{\varphi,\vartheta}-\theta_d^{\varphi,\vartheta}\right)}\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = \int_{P(T^\infty)} \mathrm{e}^{\mathrm{i}(\phi_a+\phi_b-\phi_c-\phi_d)}\,\mathrm{d}\mu_\infty$$

$\Updownarrow$ concatenate with simplex "design"

$\{|j\rangle \mid j\in\mathbb{N}\} \cup \left\{\sum_{k=1}^{\infty} \mathrm{e}^{\mathrm{i}(\vartheta k+\varphi k^2)}\,|k\rangle \mid \vartheta,\varphi\in[0,2\pi)\right\}$ forms a design on the space of tempered distributions $S(\mathbb{R})'$ (rigged continuous-variable quantum state 2-design)

Overview

Root lattices and crystal ball numbers

- Consider the root lattice A_{n-1} of $T(\mathrm{PU}(n))$
- The roots of A_{n-1} are $\mathcal{R} = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid i, j \in \{1, \ldots, n\}\}$
- The set of all points on A_{n-1} that are at most a distance s away from the origin is $s\mathcal{R}$
- The *crystal ball numbers* (OEIS:A108625) for A_{n-1} are $G_{n-1}(s) := |s\mathcal{R}|$

Theorem (Conway and Sloane 1997)

$G_{n-1}(s) = {}_3F_2(1-n, -s, n; 1, 1; 1)$

Minimal projective toric designs

- Define $P_s^{(n)} := s\mathcal{R} = \{\boldsymbol{q} - \boldsymbol{r} \mid \boldsymbol{q}, \boldsymbol{r} \in \mathbb{N}_0^n,\ \sum_{i=1}^n q_i = \sum_{i=1}^n r_i = s\}$
- $G_{n-1}(s) = |P_s^{(n)}|$
- An element $\boldsymbol{q} - \boldsymbol{r} \in P_s^{(n)}$ corresponds to a monomial $\mathrm{e}^{\mathrm{i}\sum_{j=1}^n (q_j - r_j)\phi_j}$ of degree $\leq s$ on $P(T^n)$

Minimal projective toric designs

- Define $P_s^{(n)} := s\mathcal{R} = \{\boldsymbol{q} - \boldsymbol{r} \mid \boldsymbol{q}, \boldsymbol{r} \in \mathbb{N}_0^n,\ \sum_{i=1}^n q_i = \sum_{i=1}^n r_i = s\}$
- $G_{n-1}(s) = |P_s^{(n)}|$
- An element $\boldsymbol{q} - \boldsymbol{r} \in P_s^{(n)}$ corresponds to a monomial $\mathrm{e}^{\mathrm{i}\sum_{j=1}^n (q_j - r_j)\phi_j}$ of degree $\leq s$ on $P(T^n)$

Theorem

Let $n \in \mathbb{N}$ and X a discrete, finite $P(T^n)$ t-design.

- $|X| \geq G_{n-1}(\lfloor t/2 \rfloor)$.
- *If t is even and X saturates this bound, then X is uniformly weighted.*

Overview

Group designs

- Let X be a $P(T^\infty)$ t-design and $X \cong T$
- Then $X = zT = \{(\theta z_1, \theta z_2, \ldots) \mid \theta \in [0, 2\pi)\}$ for some $z \in \mathbb{Z}^\infty$
- X is a t-design iff z satisfies (B_t *difference set*)

$$\left(\sum_{j=1}^{t} z_{a_j} = \sum_{j=1}^{t} z_{b_j}\right) \iff (\{\!\{a_j \mid j \in \{1, \ldots, t\}\}\!\} = \{\!\{b_j \mid j \in \{1, \ldots, t\}\}\!\})$$

Group designs

- Let X be a $P(T^\infty)$ t-design and $X \cong T$
- Then $X = zT = \{(\theta z_1, \theta z_2, \dots) \mid \theta \in [0, 2\pi)\}$ for some $z \in \mathbb{Z}^\infty$
- X is a t-design iff z satisfies (B_t *difference set*)

$$\left(\sum_{j=1}^{t} z_{a_j} = \sum_{j=1}^{t} z_{b_j}\right) \iff (\{\!\{a_j \mid j \in \{1, \dots, t\}\}\!\} = \{\!\{b_j \mid j \in \{1, \dots, t\}\}\!\})$$

Example

Let $z \in \mathbb{Z}^\infty$ be $z_a = t^a$. Then the group $\{(z_a\theta)_{a\in\mathbb{N}} \mid \theta \in [0, 2\pi)\}$ with its Haar measure is a $P(T^\infty)$ t-design.

Finite group designs

Definition

$z \in \mathbb{Z}_m^n$ is a B_t mod m *set of size* n if the sum mod m of any t element of z is unique.

Finite group designs

Definition

$z \in \mathbb{Z}_m^n$ is a B_t mod m *set of size* n if the sum mod m of any t element of z is unique.

Theorem

Group $P(T^n)$ *t-designs isomorphic to the cyclic group* $\mathbb{Z}_m$ *are in one-to-one correspondence with* B_t mod m *sets of size* n*.*

Finite group designs

Definition

$z \in \mathbb{Z}_m^n$ is a B_t mod m *set of size* n if the sum mod m of any t element of z is unique.

Theorem

Group $P(T^n)$ *t-designs isomorphic to the cyclic group* $\mathbb{Z}_m$ *are in one-to-one correspondence with* B_t mod m *sets of size* n*.*

Corollary

Any B_t mod m set must have size n satisfying $m \geq G_{n-1}(\lfloor t/2 \rfloor)$.

Singer sets

- Studying finite fields, Singer constructed B_t mod $\frac{(n-1)^{t+1}-1}{n-2}$ sets of size n whenever $n-1$ is a prime power.
- Hence, via these Singer sets, we have an explicit construction of $P(T^n)$ t-designs of size $\frac{(n'-1)^{t+1}-1}{n'-2}$ for all n and t, where n' is the smallest integer $\geq n$ such that $n'-1$ is a prime power.

Singer sets

- Studying finite fields, Singer constructed $B_t \bmod \frac{(n-1)^{t+1}-1}{n-2}$ sets of size n whenever $n-1$ is a prime power.
- Hence, via these Singer sets, we have an explicit construction of $P(T^n)$ t-designs of size $\frac{(n'-1)^{t+1}-1}{n'-2}$ for all n and t, where n' is the smallest integer $\geq n$ such that $n'-1$ is a prime power.
- Twirling over an overall factor of a $\mathrm{U}(1)$ $(2t)$-design, we can turn a $P(T^n)$ t-design into a T^n $(2t)$-design.
- This therefore gives explicit T^n $(2t)$-designs of size $(2t+1) \times \frac{(n'-1)^{t+1}-1}{n'-2}$ for all t and n

Sidon sets

- When $t = 2$, a B_t mod m set of size n is a *Sidon set of size n mod m*

Lower bound	Singer construction
$G_{n-1}(\lfloor t/2 \rfloor) = n(n-1)+1$	$\frac{(n'-1)^{t+1}-1}{n'-2} = n'(n'-1)+1$

Sidon sets

- When $t = 2$, a B_t mod m set of size n is a *Sidon set of size n mod m*

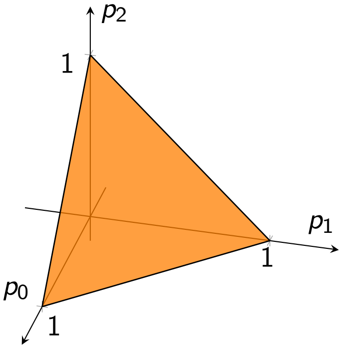
Lower bound	Singer construction
$G_{n-1}(\lfloor t/2 \rfloor) = n(n-1) + 1$	$\frac{(n'-1)^{t+1}-1}{n'-2} = n'(n'-1) + 1$

The Singer construction therefore yields *minimal* $P(T^n)$ 2-designs whenever $n - 1$ is a prime power

Overview

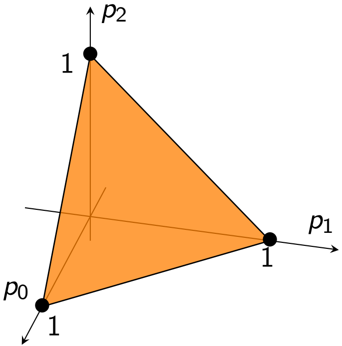
Almost minimal quantum state 2-designs

- Recall: simplex design $\times$ projective toric design yields quantum state design



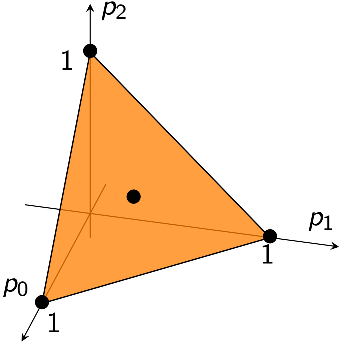
Almost minimal quantum state 2-designs

- Recall: simplex design $\times$ projective toric design yields quantum state design
- Simplex extremal points correspond to basis states $\{|1\rangle, \ldots, |d\rangle\} \subset \mathbb{C}^d$



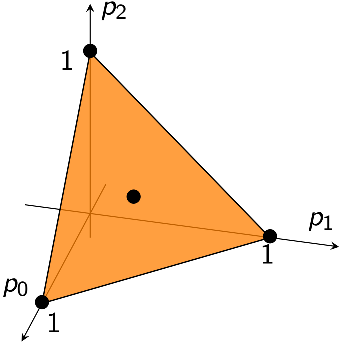
Almost minimal quantum state 2-designs

- Recall: simplex design $\times$ projective toric design yields quantum state design
- Simplex extremal points correspond to basis states $\{|1\rangle, \ldots, |d\rangle\} \subset \mathbb{C}^d$
- Simplex centroid becomes $\frac{1}{\sqrt{d}} \sum_{k=1}^{d} e^{i\phi_k} |k\rangle$, where the phases $\phi \in P(T^n)$ come from a design



Almost minimal quantum state 2-designs

- Recall: simplex design $\times$ projective toric design yields quantum state design
- Simplex extremal points correspond to basis states $\{|1\rangle, \ldots, |d\rangle\} \subset \mathbb{C}^d$
- Simplex centroid becomes $\frac{1}{\sqrt{d}} \sum_{k=1}^{d} e^{i\phi_k} |k\rangle$, where the phases $\phi \in P(T^n)$ come from a design



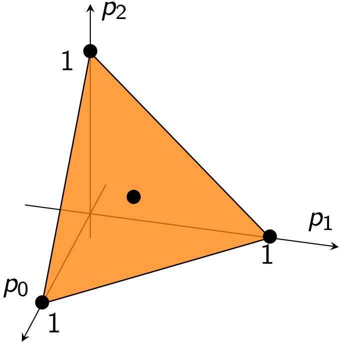
Singer's Sidon sets yield quantum state 2-designs of size $d^2 + 1$ whenever $d + 1$ is a prime power

---

These 2-designs were first constructed in (Bodmann and Haas 2016) via a totally different method

Almost minimal quantum state 2-designs

- Recall: simplex design $\times$ projective toric design yields quantum state design
- Simplex extremal points correspond to basis states $\{|1\rangle, \ldots, |d\rangle\} \subset \mathbb{C}^d$
- Simplex centroid becomes $\frac{1}{\sqrt{d}} \sum_{k=1}^{d} e^{i\phi_k} |k\rangle$, where the phases $\phi \in P(T^n)$ come from a design



> Singer's Sidon sets yield quantum state 2-designs of size $d^2 + 1$ whenever $d + 1$ is a prime power

- Recall that minimal quantum state 2-designs (SIC-POVMs) are of size d^2 (though it is still unknown if SIC-POVMs always exist)

These 2-designs were first constructed in (Bodmann and Haas 2016) via a totally different method

Overview

Summary and open questions

- We studied projective toric designs and their relationship to many other mathematical objects
- We constructed infinite families of toric and projective toric t-designs

Summary and open questions

- We studied projective toric designs and their relationship to many other mathematical objects
- We constructed infinite families of toric and projective toric t-designs
- Is our lower bound on the size of $P(T^n)$ t-designs tight? (Conjecture: when t is even)

Summary and open questions

- We studied projective toric designs and their relationship to many other mathematical objects
- We constructed infinite families of toric and projective toric t-designs
- Is our lower bound on the size of $P(T^n)$ t-designs tight? (Conjecture: when t is even)
- Must minimal designs be group designs?

Summary and open questions

- We studied projective toric designs and their relationship to many other mathematical objects
- We constructed infinite families of toric and projective toric t-designs
- Is our lower bound on the size of $P(T^n)$ t-designs tight? (Conjecture: when t is even)
- Must minimal designs be group designs?
- Connection to affine/projective planes via relationship to CS-MUBs?

Summary and open questions

- We studied projective toric designs and their relationship to many other mathematical objects
- We constructed infinite families of toric and projective toric t-designs
- Is our lower bound on the size of $P(T^n)$ t-designs tight? (Conjecture: when t is even)
- Must minimal designs be group designs?
- Connection to affine/projective planes via relationship to CS-MUBs?
- Nice simplex t-designs to generate quantum state designs via our infinite family of projective toric designs?

Summary and open questions

- We studied projective toric designs and their relationship to many other mathematical objects
- We constructed infinite families of toric and projective toric t-designs
- Is our lower bound on the size of $P(T^n)$ t-designs tight? (Conjecture: when t is even)
- Must minimal designs be group designs?
- Connection to affine/projective planes via relationship to CS-MUBs?
- Nice simplex t-designs to generate quantum state designs via our infinite family of projective toric designs?
- Concatenating designs yields a design on (effectively) the cartesion product; what about twisted products?

References

Bodmann, Bernhard G. and John Haas (Sept. 2016). "Achieving the orthoplex bound and constructing weighted complex projective 2-designs with Singer sets". In: *Linear Algebra and its Applications* 511, pp. 54–71.

Conway, J. H. and N. J. A. Sloane (Nov. 1997). "Low–dimensional lattices. VII. Coordination sequences". In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 453.1966, pp. 2369–2389.

Iosue, Joseph T. et al. (Nov. 2023). "Projective toric designs, difference sets, and quantum state designs". In: *arXiv:2311.13479*.

Iosue, Joseph T. et al. (Feb. 2024). "Continuous-Variable Quantum State Designs: Theory and Applications". In: *Phys. Rev. X* 14, p. 011013.

Kuperberg, Greg (May 2006). "Numerical Cubature from Archimedes' Hat-box Theorem". In: *SIAM Journal on Numerical Analysis* 44.3, pp. 908–935.

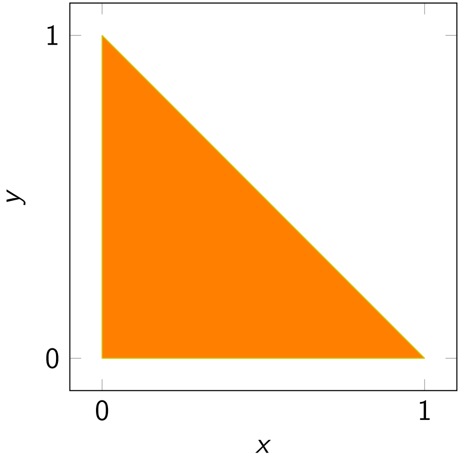
Tao, Terence and Van H Vu (2006). *Additive combinatorics*. Vol. 105. Cambridge Studies in Advanced Mathematics. Cambridge University Press.

Wootters, William K and Brian D Fields (Sept. 1989). "Optimal state-determination by mutually unbiased measurements". In: *Annals of Physics* 191.2, pp. 363–381.

Additional slides

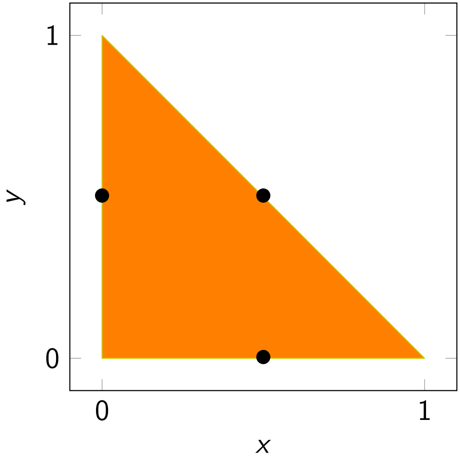
What is a t-design?

- Let $X \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$



What is a t-design?

- Let $X \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$
- Let $\mathcal{D} = \{(0,1/2),(1/2,0),(1/2,1/2)\} \subset X$



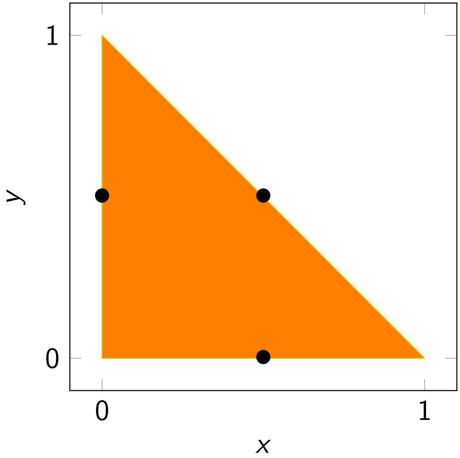
What is a t-design?

- Let $X \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$
- Let $\mathcal{D} = \{(0,1/2),(1/2,0),(1/2,1/2)\} \subset X$

$\mathcal{D}$ is a 2-*design* for X

If $g(x,y) = ax^2 + by^2 + cxy + dx + ey + f$, then

$$\frac{1}{6}\sum_{(x,y)\in\mathcal{D}} g(x,y) = \int_X g(x,y)\,\mathrm{d}x\,\mathrm{d}y$$



Fubini-Study measure

Fact

Volume integration over $\mathbb{CP}^{d-1}$ is equivalent to volume integration over $\Delta^{d-1} \times T^{d-1}$

- Consider $|p, \phi\rangle \coloneqq \sum_{n=0}^{d-1} \sqrt{p_n} \mathrm{e}^{\mathrm{i}\phi_n} |n\rangle$ for $p \in \Delta^{d-1}$ and $\phi \in \{0\} \times (\mathbb{R}/2\pi\mathbb{Z})^{d-1} \cong T^{d-1}$
- Consider $|\alpha\rangle \coloneqq \sum_{n=0}^{d-1} \alpha_n |n\rangle$ for $\alpha_n \in \mathbb{C}$, $\alpha \in S^{2d-1}$
- The natural measure on S^{2d-1} is $\prod_n \mathrm{d}^2\alpha_n$
- Under $\alpha_n \mapsto \sqrt{p_n}\mathrm{e}^{\mathrm{i}\phi_n}$, the measure becomes

$$\mathrm{d}^2\alpha_n \mapsto \mathrm{d}p_n \,\mathrm{d}\phi_n \cdot \operatorname{abs}\det \begin{pmatrix} \frac{\mathrm{e}^{\mathrm{i}\phi_n}}{2\sqrt{p_n}} & \mathrm{i}\sqrt{p_n}\mathrm{e}^{\mathrm{i}\phi_n} \\ \frac{\mathrm{e}^{-\mathrm{i}\phi_n}}{2\sqrt{p_n}} & -\mathrm{i}\sqrt{p_n}\mathrm{e}^{-\mathrm{i}\phi_n} \end{pmatrix} = \mathrm{d}p_n \,\mathrm{d}\phi_n$$

What is a design?

Definition (Cubature)

Let $X \subset \mathbb{R}^n$ and $\mathrm{d}\mu$ a measure on X. A **degree t cubature rule** for X is a finite collection of points $D \subset \mathbb{R}^n$ and a weight function $w\colon D \to \mathbb{R}$ satisfying

$$\sum_{x \in D} w(x) g(x) = \int_X g(x)\, \mathrm{d}\mu(x)$$

for any polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$ of degree t or less.

What is a design?

Definition (Cubature)

Let $X \subset \mathbb{R}^n$ and $\mathrm{d}\mu$ a measure on X. A **degree t cubature rule** for X is a finite collection of points $D \subset \mathbb{R}^n$ and a weight function $w\colon D \to \mathbb{R}$ satisfying

$$\sum_{x \in D} w(x) g(x) = \int_X g(x)\, \mathrm{d}\mu(x)$$

for any polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$ of degree t or less.

Definition (Design)

A ***t*-design** for X is a degree t cubature rule (D, w) satisfying $D \subset X$ and $\mathrm{Im}(w) \subset (0, \infty)$.

Why are designs interesting?

$X = S^d$ spherical design | $X = \mathrm{U}(d)$ unitary design | $X = \mathbb{CP}^{d-1}$ qudit design

Numerical integration	$X \subset \mathbb{R}^n$	*e.g. Stroud 1971*
Error correction	$X = S^d$	*e.g. Conway, Sloane 1999*
Randomized benchmarking	$X = \mathrm{U}(d)$	*e.g. Dankert, Cleve, Emerson, Livine 2006*
State tomography	$X = \mathbb{CP}^{d-1}$	*e.g. Scott 2006*
State distinction	$X = \mathbb{CP}^{d-1}$	*e.g. Ambainis, Emerson 2007*
Shadow tomography	$X = \mathbb{CP}^{d-1}$	*e.g. Huang, Kueng, Preskill 2020*

What is a quantum state design?

- Complex-projective space $\mathbb{CP}^{d-1} \cong S^{2d-1}/\mathrm{U}(1)$ is the set of all pure quantum states in $\mathbb{C}^d$ identified up to proportionality

Definition (Complex-projective t-design)

Let $X \subset \mathbb{CP}^{d-1}$ and $w\colon X \to (0,\infty)$. The pair (X,w) is a **complex-projective t-design** if

$$\sum_{\phi\in X} w(\phi) f(\phi) = \int_{\mathbb{CP}^{d-1}} f(\psi)\, \mathrm{d}\psi$$

for any polynomial $f(\psi)$ of degree t or less in the amplitudes and conjugate amplitudes of $|\psi\rangle$.

Generalization to arbitrary measure space

Let $X \subset \mathbb{CP}^{d-1}$ and $w\colon X \to (0, \infty)$. The pair (X, w) is a complex-projective t-design if

$$\sum_{\phi \in X} w(\phi) f(\phi) = \int_{\mathbb{CP}^{d-1}} f(\psi)\,\mathrm{d}\psi$$

for any polynomial $f(\psi)$ of degree t or less in the amplitudes and conjugate amplitudes of $|\psi\rangle$.

Let $X \subset \mathbb{CP}^{d-1}$. The measure space (X, Σ, μ) is a complex-projective t-design if

$$\int_X f(\phi)\,\mathrm{d}\mu(\phi) = \int_{\mathbb{CP}^{d-1}} f(\psi)\,\mathrm{d}\psi$$

for any polynomial $f(\psi)$ of degree t or less in the amplitudes and conjugate amplitudes of $|\psi\rangle$.

Simplex, torus, and complex-projective designs

- Consider the parameterization $|p,\phi\rangle := \sum_{n=0}^{d-1} \sqrt{p_n}\mathrm{e}^{\mathrm{i}\phi_n} |n\rangle$ for $p \in \Delta^{d-1}$ and $\phi \in (\mathbb{R}/2\pi\mathbb{Z})^{d-1} \cong T^d$
- Consider the projection $\pi\colon \mathbb{CP}^{d-1} \to \Delta^{d-1}$, $\pi(\psi) = \left(|\langle 0|\psi\rangle|^2, \ldots, |\langle d-1|\psi\rangle|^2\right)$

Simplex, torus, and complex-projective designs

- Consider the parameterization $|p,\phi\rangle := \sum_{n=0}^{d-1} \sqrt{p_n} e^{i\phi_n} |n\rangle$ for $p \in \Delta^{d-1}$ and $\phi \in (\mathbb{R}/2\pi\mathbb{Z})^{d-1} \cong T^d$
- Consider the projection $\pi \colon \mathbb{CP}^{d-1} \to \Delta^{d-1}$, $\pi(\psi) = \left(|\langle 0|\psi\rangle|^2, \dots, |\langle d-1|\psi\rangle|^2 \right)$

Fact

Volume integration over $\mathbb{CP}^{d-1}$ is equivalent to volume integration over $\Delta^{d-1} \times T^{d-1}$

Simplex, torus, and complex-projective designs

- Consider the parameterization $|p, \phi\rangle := \sum_{n=0}^{d-1} \sqrt{p_n} \mathrm{e}^{\mathrm{i}\phi_n} |n\rangle$ for $p \in \Delta^{d-1}$ and $\phi \in (\mathbb{R}/2\pi\mathbb{Z})^{d-1} \cong T^d$
- Consider the projection $\pi \colon \mathbb{CP}^{d-1} \to \Delta^{d-1}$, $\pi(\psi) = \left(|\langle 0|\psi\rangle|^2, \dots, |\langle d-1|\psi\rangle|^2 \right)$

Fact

Volume integration over $\mathbb{CP}^{d-1}$ is equivalent to volume integration over $\Delta^{d-1} \times T^{d-1}$

Theorem (Informal)

If X is a $\mathbb{CP}^{d-1}$ t-design, then $\pi(X)$ is a Δ^{d-1} t-design

Simplex, torus, and complex-projective designs

- Consider the parameterization $|p,\phi\rangle := \sum_{n=0}^{d-1} \sqrt{p_n} e^{i\phi_n} |n\rangle$ for $p \in \Delta^{d-1}$ and $\phi \in (\mathbb{R}/2\pi\mathbb{Z})^{d-1} \cong T^d$
- Consider the projection $\pi \colon \mathbb{CP}^{d-1} \to \Delta^{d-1}$, $\pi(\psi) = \left(|\langle 0|\psi\rangle|^2, \ldots, |\langle d-1|\psi\rangle|^2 \right)$

Fact

Volume integration over $\mathbb{CP}^{d-1}$ is equivalent to volume integration over $\Delta^{d-1} \times T^{d-1}$

Theorem (Informal)

If X is a $\mathbb{CP}^{d-1}$ t-design, then $\pi(X)$ is a Δ^{d-1} t-design

Theorem (Informal)

If $P \subset \Delta^{d-1}$ and $S \subset T^{d-1}$ are simplex and torus t-designs, then $P \times S$ is a $\mathbb{CP}^{d-1}$ t-design

A useful characterization of state designs

Lemma

Let $X \subset \mathbb{CP}^{d-1}$. The measure space (X, Σ, μ) is a complex-projective t-design iff

$$\int_X (|\phi\rangle\langle\phi|)^{\otimes t} \, \mathrm{d}\mu(\phi) = \int_{\mathbb{CP}^{d-1}} (|\psi\rangle\langle\psi|)^{\otimes t} \, \mathrm{d}\psi$$

A useful characterization of state designs

Lemma

Let $X \subset \mathbb{CP}^{d-1}$. The measure space (X, Σ, μ) is a complex-projective t-design iff

$$\int_X (|\phi\rangle\langle\phi|)^{\otimes t} \,\mathrm{d}\mu(\phi) = \int_{\mathbb{CP}^{d-1}} (|\psi\rangle\langle\psi|)^{\otimes t} \,\mathrm{d}\psi = \frac{\Pi_t^{(d)}}{\mathrm{Tr}\,\Pi_t^{(d)}}$$

Example (Projector onto the symmetric subspace)

- $\Pi_1^{(d)} = \mathbb{I}$
- $\Pi_2^{(d)} = \frac{1}{2}\left(\mathbb{I} \otimes \mathbb{I} + \mathrm{SWAP}\right)$