Continuous-variable quantum states designs: theory and applications

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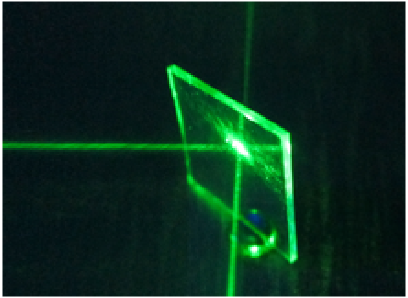
Use cases of infinite dimensions

- Continuous-variable systems are useful in technologies necessary for communication and computation
- Offers some advantages over finite-dimensional spaces
  - Continuous-parameter families of transversal gates (Eastin-Knill no-go in DV)
  - Hamiltonian-based bias-preserving gates (no-go in DV)
  - See review *V. V. Albert, arXiv:2211.05714*



Why are designs interesting?

$X = S^d$	$X = \mathrm{U}(d)$	$X = \mathbb{CP}^{d-1}$
spherical design	unitary design	qudit design

Numerical integration	$X \subset \mathbb{R}^n$	*e.g. Stroud 1971*
Error correction	$X = S^d$	*e.g. Delsarte, Goethals, Seidel 1977*
Randomized benchmarking	$X = \mathrm{U}(d)$	*e.g. Dankert, Cleve, Emerson, Livine 2006*
State tomography	$X = \mathbb{CP}^{d-1}$	*e.g. Scott 2006*
State distinction	$X = \mathbb{CP}^{d-1}$	*e.g. Ambainis, Emerson 2007*
Shadow tomography	$X = \mathbb{CP}^{d-1}$	*e.g. Huang, Kueng, Preskill 2020*

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For any $t \geq 2$, continuous-variable state/unitary t-designs **do not** *exist.*

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- Consider $\mathcal{H} = L^2(\mathbb{R})$ with (Fock) basis $\{|n\rangle \mid n \in \mathbb{N}_0\}$
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Example (Fock states plus phase states form a rigged 2-design)

$$\{|n\rangle\}_{n\in\mathbb{N}_0} \cup \left\{ |\theta_\varphi\rangle := \sum_{n\in\mathbb{N}_0} e^{i\theta n + i\varphi n^2} |n\rangle \right\}_{\theta,\varphi\in[-\pi,\pi)}$$

"Rigged" is a reference to the rigged Hilbert space prescription that is used to formalize the construction

App. 1: Continuous-variable design-based shadow tomography

- Properties of designs ensure that a relatively small number of **qubit shadows** yield a good approximation of a state for estimating observable expectation value

$$\mathop{\mathbb{E}}_{S \in \text{shadows}} S = \text{state}$$

$$S = \begin{cases} 3\,|0/1\rangle\langle 0/1| - \mathbb{I} \\ 3\,|\pm\rangle\langle\pm| - \mathbb{I} \\ 3\,|\pm\mathrm{i}\rangle\langle\pm\mathrm{i}| - \mathbb{I} \end{cases}$$

- Our phase-state + Fock-state rigged 2-designs yield CV shadows with similar guarantees

$$S = \begin{cases} (2\pi + 1)\,|\theta_\varphi\rangle\langle\theta_\varphi| - \mathbb{I} \\ (2\pi + 1)\,|n\rangle\langle n| - \mathbb{I} \end{cases}$$

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Estimate $\mathrm{Tr}(\rho\mathcal{O}_j)$ for a collection $i = j, \ldots, M$

Rigged 3-design, $N \sim \log(M)\max_j f(\mathcal{O}_j)$

Rigged 2-design, $N \sim \log(M)\max_j g(\mathcal{O}_j, \rho)$

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- Recall that a rigged t-design utilizes non-normalizable states (*i.e.* tempered distributions)
- Choose a *regularizer* R to normalize non-normalizable states while retaining important features of the design

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R projects onto a (finite-dimensional) low energy subspace of $L^2(\mathbb{R})$; *e.g.* $R = \sum_{n=0}^{d-1} |n\rangle\langle n|$

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R decays with increasing energy, but maintains support on all of $L^2(\mathbb{R})$; *e.g.* $R = \mathrm{e}^{-\beta \hat{n}}$

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$$|\theta_\varphi\rangle \propto \sum_{n\in\mathbb{N}_0} \mathrm{e}^{\mathrm{i}\theta n + \mathrm{i}\varphi n^2} |n\rangle \quad \mapsto \quad \frac{1}{\mathrm{norm}} R |\theta_\varphi\rangle \propto \sum_{n\in\mathbb{N}_0} \mathrm{e}^{-\beta n + \mathrm{i}\theta n + \mathrm{i}\varphi n^2} |n\rangle$$

App. 2: Average to entanglement fidelity

$$\bar{F}(\mathcal{E}) = \mathop{\mathbb{E}}_{\psi \in D} \langle\psi| \mathcal{E}(|\psi\rangle\langle\psi|) |\psi\rangle \quad \text{average fidelity}$$

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Finite dimensions

- $D = \mathbb{CP}^{d-1}$ or equivalently $D =$ 2-design
- $|\phi\rangle =$ maximally entangled state
- Beautiful relation *Horodecki×3 (1999)*

$$\bar{F} = \frac{dF_e + 1}{d+1}$$

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When $R = \sum_{n=0}^{d-1} |n\rangle\langle n|$, $d_R = d$
When $R = \mathrm{e}^{-\beta\hat{n}}$, $d_R = 2\,\mathrm{Tr}(\rho_\beta \hat{n}) + 1$ where ρ_β is the thermal state

Future directions

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- Find regularized rigged unitary designs

Definition (Regularized rigged unitary design)

Let $\mathcal{E}$ be an ensemble of unitaries in $\mathrm{U}(L^2(\mathbb{R}))$. $\mathcal{E}$ is an ***R*-regularized rigged unitary *t*-design** if for *all* quantum states $|\psi\rangle \in L^2(\mathbb{R})$, $\mathcal{E}\,|\psi\rangle$ is an R-regularized rigged state t-design.

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- Extend other finite dimensional design-based techniques to infinite dimensions with rigged designs (*e.g.* benchmarking continuous-variable devices)

Thanks!

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