Page curves and typical entanglement in linear optics

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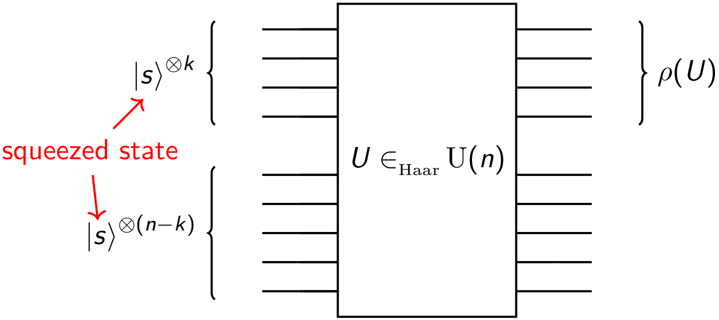
Introduction

- Gaussian Boson Sampling was one of the first experimental demonstrations of quantum supremacy
- There is a complicated relationship between entanglement and complexity
- Average and typical entanglement has been analytically computed in fermionic Gaussian states, qubits, qudits, and all of the above with certain symmetry constraints
- It has never been computed for bosonic Gaussian states!

Set up



Task: analytically compute $\mathbb{E}_{U\in\mathrm{U}(n)}$ and $\mathrm{Var}_{U\in\mathrm{U}(n)}$ of $S_\alpha(\rho(U))$ asymptotically in $n \to \infty$

Rényi-α entropy

First main result

Theorem (Rényi-2 Page curve)

Let $s \in \mathbb{R}$ *and* $r \equiv k/n \in [0,1]$. *Then, asymptotically in* $n \to \infty$,

$$\mathop{\mathbb{E}}_{U \in \mathrm{U}(n)} S_2(U) = n\alpha(s,r) - \lambda(s,r) + o(1)\,,$$

where

$$\alpha(s,r) = \sum_{\ell=1}^{\infty} \frac{\tanh^{2\ell}(2s)}{2\ell} \left(r - \frac{r^{\ell+1}}{\ell+1} \binom{2\ell}{\ell} {}_2F_1(1-\ell,\ell;\ell+2;r) \right),$$

$$\lambda(s,r) = -\frac{1}{8} \log\left(1 - 4r(1-r)\tanh^2(2s)\right),$$

where ${}_2F_1$ *is the hypergeometric function. At* $r = 1/2$, *these simplify to* $\alpha(s,1/2) = \log\cosh s$ *and* $\lambda(s,1/2) = \frac{1}{4}\log\cosh(2s)$.

First main result (graphical)

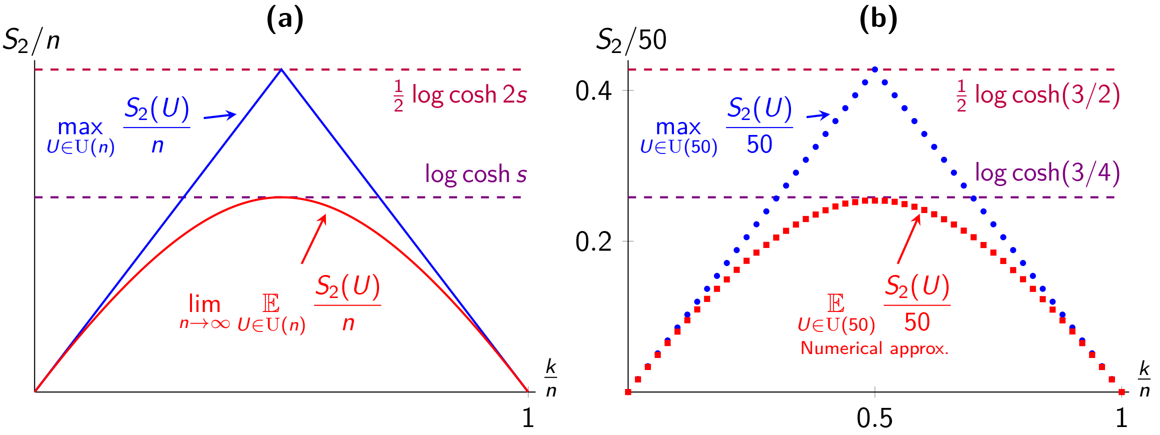


Figure: **(a)** Exact results for the Rényi-2 Page curve. **(b)** Numerical simulations of the Rényi-2 Page curve for $n = 50$ modes and squeezing $s = 3/4$. We plot the values for each $k \in \{0, 1, \ldots, 50\}$.

Typical entanglement

Definition

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S is called **strongly typical** if for any constant $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr_{U\in\mathrm{U}(n)}\left[\left|S(U) - \mathop{\mathbb{E}}_{V\in\mathrm{U}(n)} S(V)\right| < \epsilon\right] = 1.$$

Second main result

Theorem (Typical entanglement)

		$k \in \Theta(n)$	$k \in o(n)$	$k \in o(n^{1/3})$ (Fukuda 2019)
Equal	*Rényi-2*	*weak*	*strong*	*strong*
squeezing	*von Neumann*	*?*	*weak*	*strong*
Unequal	*Rényi-2*	*?*	*weak**	*strong*
squeezing	*von Neumann*	*?*	*weak**	*strong*

Table: *Rigorous results on typical entanglement in Gaussian bosonic systems. Note that "weak*" indicates that the result is not fully proven, but depends on a conjecture that we make.*

Second main result (proof)

Theorem

Let $s \in \mathbb{R}$ *and* $r \equiv k/n \in [0,1]$. *Then*

$$\lim_{n\to\infty} \operatorname*{Var}_{U\in \mathrm{U}(n)} S_2(U) = \sum_{d=2}^{\infty} \omega^{(d)} \tanh^{2d}(2s)\,(r(1-r))^d \,,$$

where $\omega^{(d)} \in \mathbb{Q}$ *is some number that depends only on* d. *In particular,* $\omega^{(2)} = 1/2$.

- Typicality comes via application of Chebyshev's inequality

Proof technique

Asymptotically in $n \to \infty$, for all $k \in \{1, \ldots, n\}$, and for all $\ell \in \mathbb{N}$, compute

$$\sum_{i_1,\ldots,i_{2\ell}=1}^{k} \sum_{i'_1,\ldots,i'_{2\ell}=1}^{k} \sum_{j_1,\ldots,j_{2\ell}=1}^{n} \sum_{j'_1,\ldots,j'_{2\ell}=1}^{n} \sum_{\sigma,\tau \in S_{2\ell}} \mathrm{Wg}(\sigma\tau^{-1}, n)$$
$$\times \delta_{i'_{2\ell},i_1} \delta_{i'_1,i_2} \delta_{i'_2,i_3} \cdots \delta_{i'_{2\ell-1},i_{2\ell}}$$
$$\times \delta_{j_1,j_2} \delta_{j'_1,j'_2} \cdots \delta_{j_{2\ell-1},j_{2\ell}} \delta_{j'_{2\ell-1},j'_{2\ell}}$$
$$\times \delta_{i_1,i'_{\sigma(1)}} \cdots \delta_{i_{2\ell},i'_{\sigma(2\ell)}}$$
$$\times \delta_{j_1,j'_{\tau(1)}} \cdots \delta_{j_{2\ell},j'_{\tau(2\ell)}},$$

where $S_{2\ell}$ denotes permutations, $\mathrm{Wg}(\sigma, n) = \frac{1}{n^{q+|\sigma|}} \prod_i (-1)^{|c_i^{(\sigma)}|-1} C_{|c_i^{(\sigma)}|-1}$, C_i is the i^{th} Catalan number, and $c_i^{(\sigma)}$ is the cyclic decomposition of the permutation σ.

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Thanks!