Continuous-variable quantum states designs: theory and applications

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arXiv:2211.05127

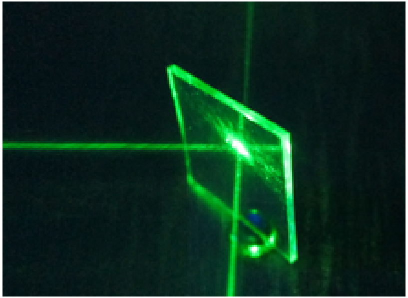
Quantum Information Processing (QIP)
08 February 2023

Overview

Overview

Use cases of infinite dimensions

- Continuous-variable systems are useful in technologies necessary for communication and computation
- Offers some advantages over finite-dimensional spaces
  - Continuous-parameter families of transversal gates (Eastin-Knill no-go in DV)
  - Hamiltonian-based bias-preserving gates (no-go in DV)
  - See review *V. V. Albert, arXiv:2211.05714*



Rough discrete & continuous variable analogies

discrete (finite)	**continuous (infinite)**
qudit	oscillator
Pauli group generated by $\{X, Z\}$	displacements generated by $\{e^{i\hat{x}}, e^{i\hat{p}}\}$
stabilizer states	Gaussian states *Gross (2006)*
Clifford group	Gaussian operations
Pauli/Clifford channels	Gaussian channels
Pauli measurements	homodyne measurements
state tomography	Wigner function
stabilizer/Clifford 2^*-design	Gaussian states/operations **NOT** 2-design

*: For prime dimensions, *Graydon et. al. (2021)*

Blume–Kohout, Turner 2011

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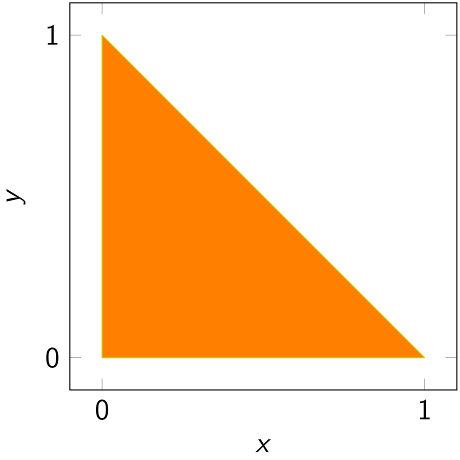
What is a t-design? $\int_{\text{hard space}}(\text{deg } t \text{ poly}) = \int_{\text{easier space}}(\text{deg } t \text{ poly})$

- Let $X \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$



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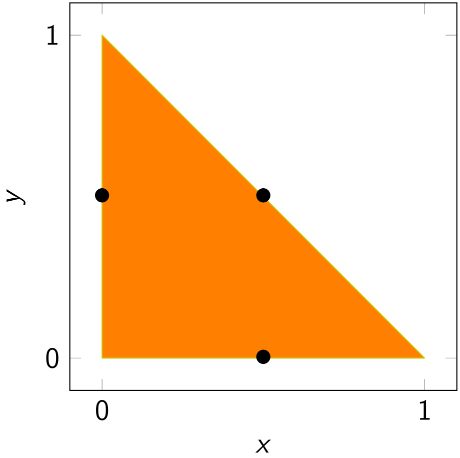
- Let $\mathcal{D} = \{(0,1/2),(1/2,0),(1/2,1/2)\} \subset X$



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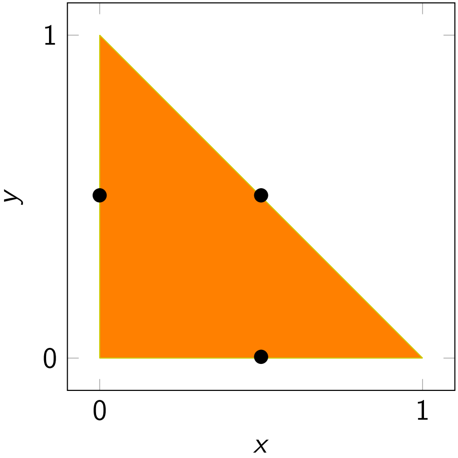
$\mathcal{D}$ is a 2-*design* for X

If $g(x,y) = ax^2 + by^2 + cxy + dx + ey + f$, then

$$\frac{1}{6}\sum_{(x,y)\in\mathcal{D}} g(x,y) = \int_X g(x,y)\,\mathrm{d}x\,\mathrm{d}y$$



Why are designs interesting?

$X = S^d$	$X = \mathrm{U}(d)$	$X = \mathbb{CP}^{d-1}$
spherical design	unitary design	qudit design

Numerical integration	$X \subset \mathbb{R}^n$	*e.g. Stroud 1971*
Error correction	$X = S^d$	*e.g. Delsarte, Goethals, Seidel 1977*
Randomized benchmarking	$X = \mathrm{U}(d)$	*e.g. Dankert, Cleve, Emerson, Livine 2006*
State tomography	$X = \mathbb{CP}^{d-1}$	*e.g. Scott 2006*
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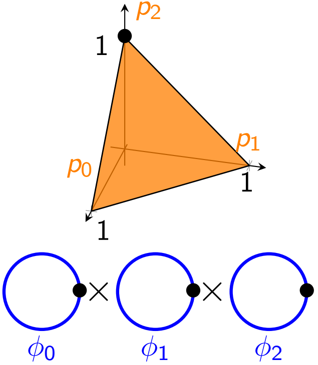
A useful characterization of quantum state designs

- Consider the parameterization $|p, \phi\rangle := \sum_{n=0}^{d-1} \sqrt{p_n}\, \mathrm{e}^{\mathrm{i}\phi_n} |n\rangle$



$\sqrt{0}\,\mathrm{e}^{\mathrm{i}(0)}|0\rangle+\sqrt{0}\,\mathrm{e}^{\mathrm{i}(0)}|1\rangle+\sqrt{1}\,\mathrm{e}^{\mathrm{i}(0)}|2\rangle$

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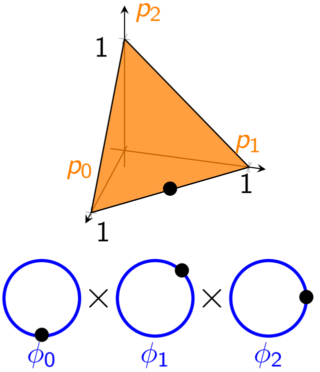

$$\sqrt{1/2}\,\mathrm{e}^{\mathrm{i}(-\pi/2)}|0\rangle + \sqrt{1/2}\,\mathrm{e}^{\mathrm{i}(\pi/4)}|1\rangle + \sqrt{0}\,\mathrm{e}^{\mathrm{i}(0)}|2\rangle$$

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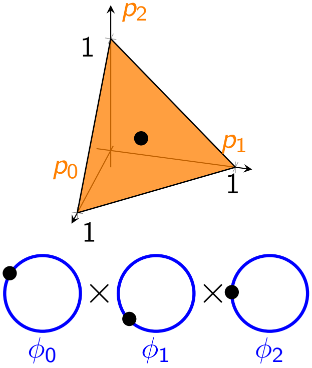

$$\sqrt{1/3}\,\mathrm{e}^{\mathrm{i}(5\pi/6)}|0\rangle + \sqrt{1/3}\,\mathrm{e}^{\mathrm{i}(-3\pi/4)}|1\rangle + \sqrt{1/3}\,\mathrm{e}^{\mathrm{i}(\pi)}|2\rangle$$

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Theorem

A quantum state t-design yields a simplex t-design.

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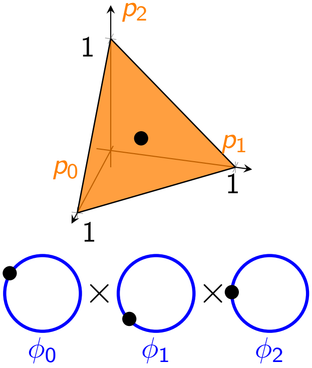
A simplex t-design and a torus t-design combine to yield a quantum state t-design.



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Overview

Non-existence of continuous-variable designs

- For any t, $\mathbb{CP}^{d-1}$ and $\mathrm{U}(d)$ t-designs exist — Seymour, Zaslavsky (1984)
- For many dimensions, the Clifford group yields a unitary 2-design — Graydon *et. al.* (2021)
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Theorem (Our work)

For any $t \geq 2$, continuous-variable state/unitary t-designs **do not** exist.

Rough intuition for the no-go theorem

- In finite dimensions, a quantum state design yields a simplex design
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- The centroid is ill-defined in the $d \to \infty$ limit

More formally, we use convergence theorems and the *Riesz Weak Compactness Theorem* to show that there does not exist a (signed or unsigned) abstract measure space satisfying the design conditions.

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How do we get around this?

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How do we get around the no-go theorem?

- Consider $\mathcal{H} = L^2(\mathbb{R})$ with (Fock) basis $\{|n\rangle \mid n \in \mathbb{N}_0\}$
- Allow ourselves to use non-normalizable states (e.g. homodyne quadrature eigenstates, GKP states, phase states)

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Example (Fock states plus phase states form a rigged 2-design)

$$\{|n\rangle\}_{n\in\mathbb{N}_0} \cup \left\{ |\theta_\varphi\rangle := \sum_{n\in\mathbb{N}_0} \mathrm{e}^{\mathrm{i}\theta n + \mathrm{i}\varphi n^2} |n\rangle \right\}_{\theta,\varphi\in[-\pi,\pi)}$$

- Phase states give us the required non-normalizable centroids!

"Rigged" is a reference to the rigged Hilbert space prescription that is used to formalize the construction

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App. 1: Continuous-variable design-based shadow tomography

- Properties of designs ensure that a relatively small number of **qubit shadows** yield a good approximation of a state for estimating observable expectation value

$$S = \begin{cases} 3\,|0/1\rangle\langle 0/1| - \mathbb{I} \\ 3\,|\pm\rangle\langle \pm| - \mathbb{I} \\ 3\,|\pm \mathrm{i}\rangle\langle \pm \mathrm{i}| - \mathbb{I} \end{cases}$$

$$\mathop{\mathbb{E}}_{S \in \text{shadows}} S = \text{state}$$

- Our phase-state + Fock-state rigged 2-designs yield CV shadows with similar guarantees

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Estimate $\mathrm{Tr}(\rho\mathcal{O}_j)$ for a collection $i = j, \ldots, M$

Rigged 3-design, $N \sim \log(M)\max_j f(\mathcal{O}_j)$

Rigged 2-design, $N \sim \log(M)\max_j g(\mathcal{O}_j, \rho)$

App. 2: Regularized rigged designs

- Recall that a rigged t-design utilizes non-normalizable states (*i.e.* tempered distributions)
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R projects onto a (finite-dimensional) low energy subspace of $L^2(\mathbb{R})$; *e.g.* $R = \sum_{n=0}^{d-1} |n\rangle\langle n|$

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$$|\theta_\varphi\rangle \propto \sum_{n\in\mathbb{N}_0} \mathrm{e}^{\mathrm{i}\theta n+\mathrm{i}\varphi n^2} |n\rangle \;\mapsto\; \frac{1}{\mathrm{norm}} R\,|\theta_\varphi\rangle \propto \sum_{n\in\mathbb{N}_0} \mathrm{e}^{-\beta n+\mathrm{i}\theta n+\mathrm{i}\varphi n^2} |n\rangle$$

App. 2: Average to entanglement fidelity

$$\bar{F}(\mathcal{E}) = \mathop{\mathbb{E}}_{\psi \in D} \langle\psi| \mathcal{E}(|\psi\rangle\langle\psi|) |\psi\rangle \quad \text{average fidelity}$$

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- $D = \mathbb{CP}^{d-1}$ or equivalently $D =$ 2-design
- $|\phi\rangle =$ maximally entangled state
- Beautiful relation *Horodecki×3 (1999)*

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When $R = \sum_{n=0}^{d-1} |n\rangle\langle n|$, $d_R = d$
When $R = \mathrm{e}^{-\beta \hat{n}}$, $d_R = 2\,\mathrm{Tr}(\rho_\beta \hat{n}) + 1$ where ρ_β is the thermal state

Overview

Summary

- Continuous-variable (*e.g.* oscillator) state and unitary t-designs do not exist for any $t \geq 2$
- The reason they do not exist is due to state normalization
- Remove normalization (*i.e.* go to *rigged Hilbert space*) to generate *rigged designs*

Summary

- Continuous-variable (*e.g.* oscillator) state and unitary t-designs do not exist for any $t \geq 2$
- The reason they do not exist is due to state normalization
- Remove normalization (*i.e.* go to *rigged Hilbert space*) to generate *rigged designs*
- Rigged designs are POVMs *plus a little extra* — allows for shadow tomography
- Regularized rigged designs apply soft-energy cutoff — allows for notions of average fidelity

Future directions

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- Find regularized rigged unitary designs

Definition (Regularized rigged unitary design)

Let $\mathcal{E}$ be an ensemble of unitaries in $\mathrm{U}(L^2(\mathbb{R}))$. $\mathcal{E}$ is an ***R*-regularized rigged unitary *t*-design** if for *all* quantum states $|\psi\rangle \in L^2(\mathbb{R})$, $\mathcal{E}\,|\psi\rangle$ is an R-regularized rigged state t-design.

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- Extend other finite dimensional design-based techniques to infinite dimensions with rigged designs (*e.g.* benchmarking continuous-variable devices)

Thanks!

Kunal Sharma

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Victor V. Albert

Additional slides

Sketch of the no-go theorem for $t = 2$

- Projecting to the infinite-dimensional simplex, we find that if a CV 2-design exists, then there exists a σ-finite measure space (X, Σ, μ) and a sequence $(p_i)_{i\in\mathbb{N}_0}$ of measurable maps $p_i\colon X \to [0,1]$ satisfying
  - $\sum_{i\in\mathbb{N}_0} p_i(x) = 1$ for almost all $x \in X$, and
  - $\int_X p_a(x)p_b(x)\,\mathrm{d}\mu(x) = \frac{1}{2}(1+\delta_{ab})$ for any $a, b \in \mathbb{N}_0$
- Riesz Weak Compactness Theorem: there exists a q such that for all $h \in L^2(X)$, $\lim_{a\to\infty} \int_X p_a h\,\mathrm{d}\mu = \int_X qh\,\mathrm{d}\mu$
- Lebesgue Dominated Convergence Theorem: $\lim_{a\to\infty} \int_X p_a p_b p_c\,\mathrm{d}\mu = 0$; implies that $q = 0$ a.e.
- Therefore, $\lim_{a\to\infty} \int_X p_a p_b\,\mathrm{d}\mu = 0 \neq \lim_{a\to\infty} \frac{1}{2}(1+\delta_{ab}) = \frac{1}{2}$

Regularized rigged unitary design

Definition ($\mathrm{U}(d)$ t-design)

Let $\mathcal{E}$ be an ensemble of unitaries in $\mathrm{U}(d)$. $\mathcal{E}$ is a **unitary *t*-design** if for *all* quantum states $|\psi\rangle \in \mathbb{CP}^{d-1}$, $\mathcal{E}\,|\psi\rangle$ is a quantum state t-design.

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Discrete to continuous-variable

Discrete variable		**Continuous variable**
$\Pi_t^{(d)} : (\mathbb{C}^d)^{\otimes t} \to (\mathbb{C}^d)^{\otimes t}$	$d \to \infty$ →	$\Pi_t : L^2(\mathbb{R})^{\otimes t} \to L^2(\mathbb{R})^{\otimes t}$

Fact ↓

$$\int_{\mathbb{CP}^{d-1}} (|\psi\rangle\langle\psi|)^{\otimes t} \, \mathrm{d}\psi \propto \Pi_t^{(d)}$$

Definition ↓ ($\mathbb{CP}^{d-1}$ *t-design*)   **Definition** ↓ (*CV t-design*)

$(X \subset \mathbb{CP}^{d-1}, \Sigma, \mu)$ s.t. $\int_X (|\psi\rangle\langle\psi|)^{\otimes t} \, \mathrm{d}\mu(\psi) \propto \Pi_t^{(d)}$	$d \to \infty$ →	$(X \subset L^2(\mathbb{R}), \Sigma, \mu)$ s.t. $\int_X (|\psi\rangle\langle\psi|)^{\otimes t} \, \mathrm{d}\mu(\psi) \propto \Pi_t$

Continuous-variable to rigged

Continuous variable

Rigged

$\Pi_t \colon L^2(\mathbb{R})^{\otimes t} \to L^2(\mathbb{R})^{\otimes t}$

$\Pi_t \colon S(\mathbb{R})^{\otimes t} \to S(\mathbb{R})^{\otimes t}$

Definition (*CV t-design*)

$(X \subset L^2(\mathbb{R}), \Sigma, \mu)$ s.t. $\int_X (|\psi\rangle\langle\psi|)^{\otimes t} \, \mathrm{d}\mu(\psi) \propto \Pi_t$

Definition (*Rigged t-design*)

$(X \subset S(\mathbb{R})', \Sigma, \mu)$ s.t. $\int_X (|\chi\rangle\langle\chi|)^{\otimes t} \, \mathrm{d}\mu(\chi) \propto \Pi_t$