Page curves and typical entanglement in linear optics

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Introduction

Entanglement is a key feature of quantum physics and can be used as a resource to complete various tasks, such as teleportation, key distribution, dense coding, and many others. Studying average and typical entanglement is necessary for learning about the useful part of entanglement and what utility random states have. In this work, we consider the average and typical entanglement of Gaussian Boson Sampling output states.

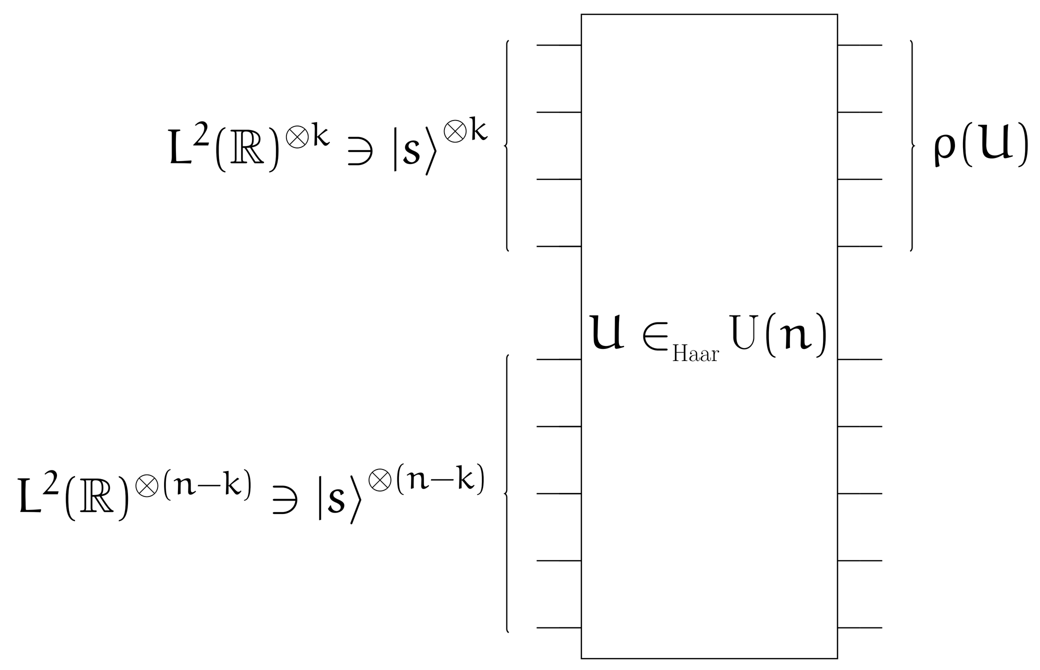

Figure 1: Pictorial representation of the setup we consider.

To begin, n harmonic oscillator modes are initialized in a product states of squeezed vacuum states $|s\rangle \in L^2(\mathbb{R})$ with squeezing strength s. Then, a Haar random passive (energy-conserving) Gaussian unitary U is applied to the n modes. Note that the group of passive Gaussian unitaries is isomorphic to $\mathrm{Sp}(2n) \cap \mathrm{O}(2n) \cong \mathrm{U}(n)$. Finally, we consider the Rényi-2 entanglement entropy, $S_2(U) = -\log \operatorname{Tr} \rho(U)^2$, of the final reduced state on $1 \le k \le n$ modes, $\rho(U)$.

Average entanglement

We study the average entanglement in a subsystem of k modes, $\mathbb{E}_{U \in U(n)} S_2(U)$. Viewed as a function of k, this quantity is known as the **Page curve**. Using the Rényi-2 Page curve, we can also upper and lower bound the von Neumann Page curve.

Theorem (Rényi-2 Page curve)

Let $s \in \mathbb{R}$ and $r \equiv k/n \in [0,1]$. Then, asymptotically in $n \to \infty$,

$$\mathop{\mathbb{E}}_{U \in U(n)} S_2(U) = n\alpha(s,r) - \lambda(s,r) + o(1),$$

where

$$\alpha(s,r) = \sum_{\ell=1}^{\infty} \frac{\tanh^{2\ell}(2s)}{2\ell}\left(r - \frac{r^{\ell+1}}{\ell+1}\binom{2\ell}{\ell}\,{}_2F_1(1-\ell,\ell;\ell+2;r)\right),$$

$$\lambda(s,r) = -\frac{1}{8}\log\left(1 - 4r(1-r)\tanh^2(2s)\right),$$

where ${}_2F_1$ is the hypergeometric function. At $r = 1/2$, these simplify to $\alpha(s,1/2) = \log\cosh s$ and $\lambda(s,1/2) = \frac{1}{4}\log\cosh(2s)$.

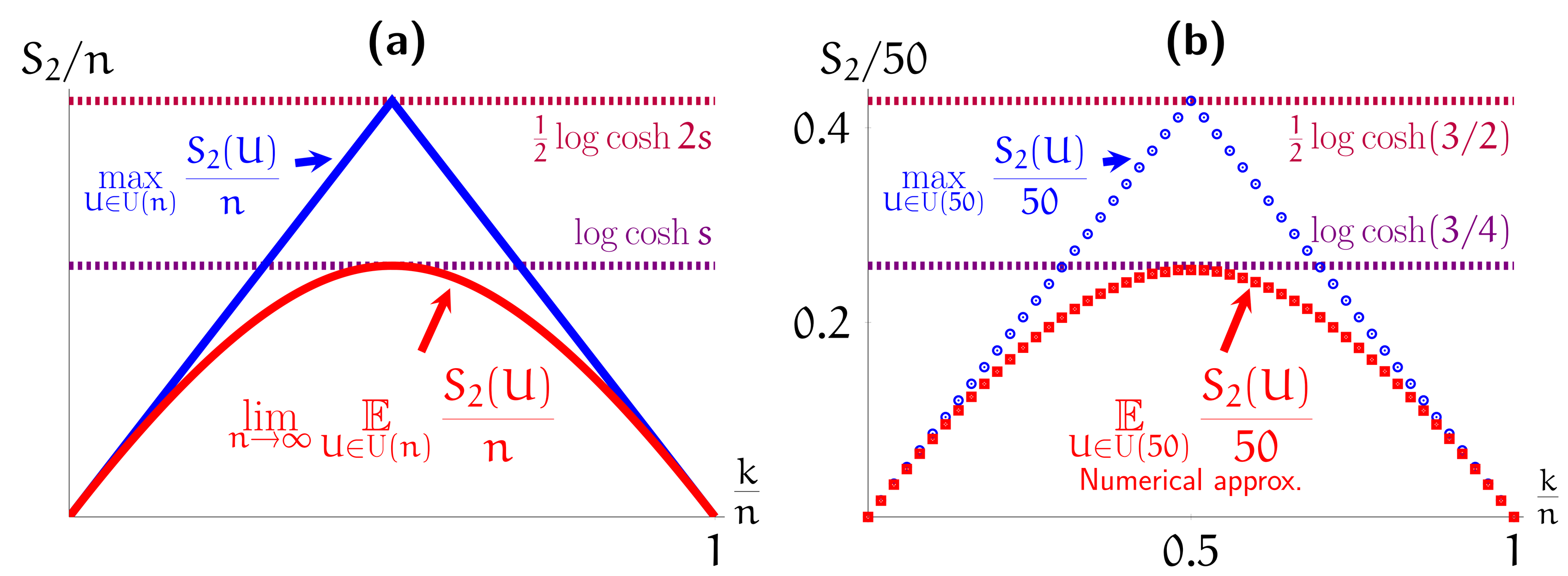

Figure 2: (a) Exact results for the Rényi-2 Page curve. (b) Numerical simulations of the Rényi-2 Page curve for $n = 50$ modes and squeezing $s = 3/4$. We plot the values for each $k \in \{0, 1, \ldots, 50\}$.

$\alpha(s,r)$ can be written as $\alpha(s,r) = \sum_{\ell=1}^{\infty} \frac{\tanh^2(2s)}{2\ell} G_\ell(r)$, where $G_\ell(r) := r - f_\ell(r)$ and $f_\ell(r)$ is a polynomial of degrees $\ell+1$ through 2ℓ in r. Polynomials $G_\ell(r)$ of this form are uniquely determined by the requirement that $G_\ell(r) = G_\ell(1-r)$, which ensures that the Rényi-2 entropy of a subsystem is equal to that of its complement since we are considering pure states. It is from this requirement that we ultimately derive the Page curve. We show that the resulting $G_\ell(r)$ can be understood as a good approximation to $m(r) := \min(r, 1-r)$ from below. Indeed, the approximation is especially good near the endpoints $r = 0$ and $r = 1$, where the first ℓ derivatives of $G_\ell(r)$ match those of $m(r)$. As $\ell \to \infty$, the approximation becomes better and better such that $\lim_{\ell\to\infty} G_\ell(r) = m(r)$. This provides an interpretation of the derived form of the Page curve. The strength of the squeezing s determines the weight that the Page curve has on the ℓ^{th} approximation to $m(r)$. For small squeezing, only low order approximations contribute, with the most dominant contribution being the parabolic shape $G_1(r) = r(1-r)$. When the squeezing is increased, there is more contribution from higher order approximations, giving the Page curve more of the triangle shape of $m(r)$. We see a manifestation of this interpretation as

$$\lim_{s\to 0} \frac{1}{s^2}\alpha(s,r) = 2r(1-r), \qquad \lim_{s\to\infty} \frac{1}{s}\alpha(s,r) = 2\min(r, 1-r).$$

Typical entanglement

Typicality is of interest because it characterizes the applicability of statistical averages. In order to quantify the deviation from average, we consider two measures of deviation corresponding to multiplicative and additive distance. If the multiplicative distance between a quantity and its average vanishes in the thermodynamic limit, then that quantity is called **weakly typical**. If the additive distance vanishes in this limit, then that quantity is called **strongly typical**.

Definition

Let S be a nonnegative random variable on the unitary group $U(n)$, and denote its value at $U \in U(n)$ by $S(U)$. S is called *weakly typical* if for any constant $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr_{U\in U(n)}\left[\left|\frac{S(U)}{\mathbb{E}_{V\in U(n)} S(V)} - 1\right| < \epsilon\right] = 1. \tag{1}$$

S is called *strongly typical* if for any constant $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr_{U\in U(n)}\left[\left|S(U) - \mathop{\mathbb{E}}_{V\in U(n)} S(V)\right| < \epsilon\right] = 1. \tag{2}$$

Theorem (Typical entanglement)

		$k \in \Theta(n)$	$k \in o(n)$	$k \in o(n^{1/3})$ *[2]*
Equal squeezing	*Rényi-2*	*weak*	*strong*	*strong*
	von Neumann	*?*	*weak*	*strong*
Unequal squeezing	*Rényi-2*	*?*	*weak**	*strong*
	von Neumann	*?*	*weak**	*strong*

Table 1: Rigorous results on typical entanglement in Gaussian bosonic systems. Note that "weak*" indicates that the result is not fully proven, but depends on a conjecture that we make.

The typical entanglement proof again crucially relies on the symmetry $k \mapsto n-k$ of the entanglement entropy. Specifically, we derive the functional form of the variance $\mathrm{Var}_{U\in U(n)} S_2(U)$ using the symmetry and show that the variance is asymptotically independent of the number of modes. Then, we utilize the variance in Chebyshev's inequality to prove typicality.

Theorem

Let $s \in \mathbb{R}$ and $r \equiv k/n \in [0,1]$. Then

$$\lim_{n\to\infty} \mathop{\mathrm{Var}}_{U\in U(n)} S_2(U) = \sum_{d=2}^{\infty} \omega^{(d)} \tanh^{2d}(2s)\,(r(1-r))^d,$$

where $\omega^{(d)} \in \mathbb{Q}$ is some number that depends only on d. In particular, $\omega^{(2)} = 1/2$.

References

[1] J. T. Iosue, A. Ehrenberg, D. Hangleiter, A. Deshpande, and A. V. Gorshkov, "Page curves and typical entanglement in linear optics", arXiv:2209.06838 (2022).

[2] M. Fukuda and R. Koenig, "Typical entanglement for Gaussian states", Journal of Mathematical Physics **60**, 112203 (2019).